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# A new integrable system on $S^2$ with the second integral quartic in the momenta

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**Abstract**
We construct a new integrable system on the sphere $S^2$ with an additional integral of fourth order in the momenta using standard machinery of the reflection equation theory. At the special values of parameters, this system coincides with the Kowalevski–Goryachev–Chaplygin system.

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## 1. Introduction

The description of all the natural Hamiltonian systems on closed surfaces admitting integrals polynomial in the momenta is a classical problem [1]. According to Maupertuis's principle, an integrable natural Hamiltonian system immediately gives a family of integrable geodesics [2]. If the integral of the system is polynomial in momenta, the integrals of the geodesic are also polynomial of the same degree.

For the natural Hamiltonian systems on closed surfaces with polynomial in momenta integrals of degree 1 or 2, there exists a complete description and classification [3]. According to [4], a geodesic on surfaces of genus greater than two cannot admit a nontrivial integral polynomial in the momenta. Then, an orientable surface admitting such geodesics must be the sphere or the torus.

The motion on a sphere is also related to other different physical systems [5, 6]. Thus, investigation of integrable natural systems on the sphere with nontrivial integrals polynomial in the momenta is an interesting mathematical and physical problem.

There are two families of natural Hamiltonian systems on a sphere with a cubic additional integral of motion. The systems from these families are closely related with the spherical top [7] and with the Goryachev–Chaplygin top [8].

The Kowalevski top is an example of a conservative system on the sphere which possesses an integral of degree four in the momenta [9]. Later Goryachev [10] and Chaplygin [11] found conservative systems on the sphere, which are generalizations of the Kowalevski system. Recently, these results were extended in [12].

The aim of this paper is to consider a new integrable system on the sphere possessing integrals of second and fourth order in the momenta using the reflection equation theory [13].

## 2. The Kowalevski–Goryachev–Chaplygin system

Let us briefly recall the construction of the Lax matrices in the framework of the reflection equation theory. Let $2 \times 2$ matrix $T(\lambda)$ defines the representation of the Sklyanin algebra on the space $M$

$$\left\{ \overset{1}{T}(\lambda), \overset{2}{T}(\nu) \right\} = \left[ r(\lambda - \nu), \overset{1}{T}(\lambda)\overset{2}{T}(\nu) \right]. \tag{2.1}$$

Here, $r$ is a classical $r$-matrix and $\overset{1}{T}(\lambda) = T(\lambda) \otimes \mathrm{Id}, \overset{2}{T}(\nu) = \mathrm{Id} \otimes T(\nu)$.

One of the main properties of the Sklyanin algebra is that for *any* non-dynamical matrices $\mathcal{K}$ and for some *special* dynamical matrices $\mathcal{K}$ coefficients of the trace of the Lax matrix $\mathrm{L}(\lambda) = \mathcal{K}T(\lambda)$ give rise to the commutative subalgebra in $C(M)$

$$\{\mathrm{tr}\, \mathcal{K}T(\lambda), \mathrm{tr}\, \mathcal{K}T(\mu)\} = 0, \tag{2.2}$$

(see [17] and references within). Thus, coefficients of polynomial $\mathrm{tr}\, L(\lambda)$ may be interpreted as integrals of motion for an integrable system on the phase space $M$.

According to [13], if $\mathcal{K}_\pm(\lambda)$ are solutions of the reflection equation

$$\left\{ \overset{1}{\mathcal{K}}(\lambda), \overset{2}{\mathcal{K}}(\nu) \right\} = \left[ r(\lambda - \nu), \overset{1}{\mathcal{K}}(\lambda)\overset{2}{\mathcal{K}}(\nu) \right] + \overset{1}{\mathcal{K}}(\lambda) r(\lambda + \nu) \overset{2}{\mathcal{K}}(\nu) - \overset{2}{\mathcal{K}}(\nu) r(\lambda + \nu) \overset{1}{\mathcal{K}}(\lambda),$$

then coefficients of the trace of the Lax matrix

$$L(\lambda) = \mathcal{K}_-(\lambda) T(\lambda - \rho) \mathcal{K}_+(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^t(-\lambda - \rho) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.3}$$

give rise to another commutative subalgebra in $C(M)$

$$\{\mathrm{tr}\, L(\lambda), \mathrm{tr}\, L(\nu)\} = 0. \tag{2.4}$$

In (2.3), the superscript $t$ stands for matrix transposition, matrix $T(\lambda)$ satisfies the Sklyanin algebra (2.1) and commutes with $\mathcal{K}(\lambda)$.

It is easy to see that the key ingredient of the proposed scheme is the matrix $T(\lambda)$, which defines three Lax matrices $T(\lambda)$, $L(\lambda)$ and $L(\lambda)$ for three different integrable systems on the common phase space $M$ [17].

As an example let us consider the Kowalewski–Chaplygin–Goryaschev system on the sphere $S^2 = \{x \in \mathbb{R}^3, |x| = a\}$. Entries of the vector $x$ and angular momentum vector $J = p \times x$ are coordinates on the phase space $T^*S^2$ with the following Poisson brackets:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \qquad \{x_i, x_j\} = 0, \tag{2.5}$$

where $\varepsilon_{ijk}$ is the totally skew-symmetric tensor. The Casimir functions of the brackets (2.5)

$$A = \sum_{i=1}^{3} x_i^2 = a^2, \qquad B = \sum_{i=1}^{3} x_i J_i = 0 \tag{2.6}$$

are in the involution with any function on $T^*S^2$. The phase space $T^*S^2$ is four-dimensional symplectic manifold. So, for the Liouville integrability of the corresponding equations of motion it is enough to find two functionally independent integrals of motion.

The Hamilton function for the Kowalewski–Chaplygin–Goryaschev system is equal to

$$H = J_1^2 + J_2^2 + 2J_3^2 + \rho J_3 + b x_1 + c\left(x_1^2 - x_2^2\right) + \frac{\delta}{x_3^2}, \qquad \rho, b, c, \delta \in \mathbb{R}. \tag{2.7}$$

The corresponding additional integral of motion are fourth-order polynomial in the momenta [9–11].

The $2 \times 2$ Lax matrix (2.3) for this system was constructed in [14, 15], whereas the corresponding separation of variables is discussed in [16]. The starting point is the Lax matrix for the symmetric Neumann system

$$T(\lambda)=\begin{pmatrix}\lambda^2-2J_3\lambda-J_1^2-J_2^2-\frac{\delta}{x_3^2} & \lambda(x_1+\mathrm{i}x_2)-x_3(J_1+\mathrm{i}J_2)\\ \lambda(x_1-\mathrm{i}x_2)-x_3(J_1-\mathrm{i}J_2) & x_3^2\end{pmatrix}, \tag{2.8}$$

which defines representation of the Sklyanin algebra (2.1) on $T^*S^2$, associated with the standard rational $r$-matrix

$$r(\lambda-\nu)=\frac{\mathrm{i}}{\lambda-\nu}\Pi,\qquad \Pi=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

If we substitute matrix $T(\lambda)$ (2.8) and two constant solutions of the reflection equations

$$\mathcal{K}_+=\begin{pmatrix}b_1\lambda+b_0 & \lambda\\ 0 & -b_1\lambda+b_0\end{pmatrix},\qquad \mathcal{K}_-=\begin{pmatrix}d_1\lambda+d_0 & 0\\ \lambda & -d_1\lambda+d_0\end{pmatrix} \tag{2.9}$$

into definition (2.3), one gets the Lax matrix $L(\lambda)$ for the Kowalevsky–Goryachev–Chaplygin gyrostat. In this case, the trace of the Lax matrix $L(\lambda)$ (2.3)

$$\operatorname{tr} L(\lambda)=\lambda^6-2\widetilde{H}\lambda^4+\widetilde{K}\lambda^2+2b_0d_0(A\rho^2-\delta)$$

is a generating function of the integrals of motion $\widetilde{H}$ and $\widetilde{K}$ on $T^*S^2$, which are in the involution $\{\widetilde{H},\widetilde{K}\}=0$ according to (2.4).

After a suitable canonical transformation of variables and exchange of parameters the Hamilton function

$$\begin{aligned}\widetilde{H}=J_1^2+J_2^2+2J_3^2+2\rho J_3+\rho^2-b_1d_1a-\mathrm{i}(b_0+d_0-(b_1+d_1)(2J_3+\rho))x_1\\ -(b_0-d_0-(b_1-d_1)(2J_3+\rho))x_2+(\mathrm{i}(b_1+d_1)J_1+(b_1-d_1)J_2)x_3+\frac{\delta}{x_3^2}\end{aligned}$$

coincides with the original Hamilton function $H$ (2.7), see [16] for details.

In the next section, we consider the generalization of the matrix $T(\lambda)$ (2.8) in order to construct a new integrable system.

## 3. Deformation of the Kowalevski system

Let us consider the deformation of the matrix $T(\lambda)$ (2.8) proposed in [8]. It is the Lax matrix for the generalized Lagrange system

$$T_\alpha(\lambda)=\begin{pmatrix}A & B\\ B^* & D\end{pmatrix}(\lambda),$$

where

$$\begin{aligned}A(\lambda)&=\lambda^2-2\lambda\alpha J_3+(\alpha^2-f(x_3))J_3^2-J_1^2-J_2^2-g(x_3),\\ B(\lambda)&=(x_1+\mathrm{i}x_2)m(x_3)\lambda+J_3(x_1+\mathrm{i}x_2)\ell(x_3)+(J_1+\mathrm{i}J_2)n(x_3),\\ D(\lambda)&=-n(x_3)^2.\end{aligned} \tag{3.1}$$

Here, $\alpha\in\mathbb{R}$ and $f,g,m,n$ and $\ell$ are some functions of $x_3$ and of the single nontrivial Casimir $a=\sqrt{x_1^2+x_2^2+x_3^2}$ (2.6).

According to [8], at the special values of $\alpha,f,g,m,n$ and $\ell$ matrix $T_\alpha(\lambda)$ satisfies deformations of the Sklyanin brackets (2.1), which inherit the property (2.2). In these cases, matrix $T_\alpha$ defines new Lax matrices $L_\alpha(\lambda)=\mathcal{K}T_\alpha(\lambda)$ for five integrable systems on $T^*S^2$ with additional cubic integrals of motion.

In this paper, the same matrix $T_\alpha(\lambda)$ will be used in the machinery of the reflection equation theory. Substituting $T_\alpha(\lambda)$ (3.1) into the generic definition (2.3) one gets a $2\times 2$ matrix $L_\alpha(\lambda)$ with the following trace:

$$\operatorname{tr} L_\alpha(\lambda) = -\lambda^6 + 2H_\alpha\lambda^4 + K_\alpha\lambda^2 + 2F_\alpha. \tag{3.2}$$

If this matrix $L_\alpha(\lambda)$ inherits the property (2.4) then it is a Lax matrix for some new integrable system with integrals of motion $H_\alpha$, $K_\alpha$ and $F_\alpha$ (3.2) in the involution.

In the generic case, equation (2.4) gives rise to a system of equations on $\alpha$ and $f, g, m, n, \ell$. Let us solve these equation in the simplest case. If $d_1 = b_1$, $d_0 = b_0$ and $\rho = 0$ then

$$\begin{aligned}
H_\alpha &= J_1^2 + J_2^2 + (f+\alpha^2)J_3^2 - 2b_0x_1m + 2b_1(nJ_1 + (\ell + 2m\alpha)x_1J_3) + b_1^2\big(m^2\big(a^2 - x_3^2\big) + n^2\big),\\
K_\alpha &= \big(J_1^2 + J_2^2 + (f-\alpha^2)J_3^2 + g + 2b_1(x_1J_3\ell + nJ_1)\big)^2 + 8\alpha b_1b_0n^2J_3\\
&\quad + 2b_1^2\big(J_3^2(n^2(f-\alpha^2) - \big(a^2 - x_3^2\big)\ell^2\big) - 2\ell n(x_1J_1 - x_2J_2)J_3 + \big(g + 2J_2^2\big)n^2\big)\\
&\quad + 4b_0\big(x_1(2\alpha\ell - (f-\alpha^2)m)J_3^2 + 2\alpha nJ_1J_3 - x_1m\big(g + J_1^2 + J_2^2\big)\big)\\
&\quad + 2b_0^2\big(m^2\big(a^2 - x_3^2\big)^2 - n^2\big)\\
F_\alpha &= b_0^2\big((\alpha^2 - f)n^2 - 2x_3\ell n + \big(a^2 - x_3^2\big)\ell^2\big)J_3^2 - b_0^2n^2g.
\end{aligned} \tag{3.3}$$

Here, we omit the dependence of $x_3$ in $f, g, m, n$ and $\ell$ for brevity.

Functions $H_\alpha, F_\alpha$ are the second-order polynomials in the momenta whereas $K_\alpha$ is a quartic polynomial. Moreover, it is easy to see that function $F_\alpha$ depends on variables $x_3$ and $J_3$ only. So, if we want to consider integrable systems different from the generalized Lagrange system we have to put $F_\alpha = \text{const}$. It leads to the following expressions of the functions $f(x_3)$ and $g(x_3)$:

$$f(x_3) = \alpha^2 - \frac{2\ell(x_3)x_3}{n(x_3)} + \frac{\ell(x_3)^2\big(a^2 - x_3^2\big)}{n(x_3)^2}, \qquad g(x_3) = \frac{d}{n(x_3)^2}, \tag{3.4}$$

where $d \in \mathbb{R}$.

**Theorem 1.** *If $f(x_3)$ and $g(x_3)$ are given by (3.4) then function $F_\alpha$ in (3.2), (3.3) is a constant*

$$F_\alpha = -b_0^2 d,$$

*while two remaining functions $H_\alpha$ and $K_\alpha$ on $T^*S^2$ are in the involution*

$$\{H_\alpha, K_\alpha\} = 0 \tag{3.5}$$

*with respect to the brackets (2.5) if and only if*

$$n(x_3, a) = c_1 \sin\left(\alpha \arctan\left(\frac{x_3}{\sqrt{a^2 - x_3^2}}\right)\right) + c_2 \cos\left(\alpha \arctan\left(\frac{x_3}{\sqrt{a^2 - x_3^2}}\right)\right). \tag{3.6}$$

*Here, $\alpha, c_1, c_2$ are arbitrary parameters and all the other functions in (3.1), (3.3) are equal to*

$$\begin{aligned}
&\alpha = 0, \qquad m = 0, \qquad \ell = n\frac{\sqrt{x_3^2 - a^2} - x_3\big(\ln\big(x_3 + \sqrt{x_3^2 - a^2}\big) + c_3\big)}{\big(x_3^2 - a^2\big)\big(\ln\big(x_3 + \sqrt{x_3^2 - a^2}\big) + c_3\big)},\\
&\alpha \neq 0, \qquad m = -\frac{n'}{\alpha}, \qquad \ell = \frac{(\alpha^2 n - x_3 n')n}{\big(x_3^2 - a^2\big)n'},
\end{aligned} \tag{3.7}$$

*where $n' = \frac{\partial n(x_3, a)}{\partial x_3}$.*

**Proof.** Insert integrals $H_\alpha$ and $K_a$ (3.3) into equation (3.5), which has to be satisfied identically with respect to three independent variables $J_{1,2}$ and $x_1$. Two dependent variables $J_3$ and $x_2$ have to be removed by using the Casimir functions $A$ and $B$.

At $\alpha \neq 0$, this gives a system of algebraic equations for $\ell$, $m$ and one differential equation for $n(x_3)$, which should be solved. Substituting solutions (3.7) of the algebraic equations into the differential equation for $n(x_3)$ one gets

$$\left(x_3^2 - a^2\right)n''(x_3) + x_3 n'(x_3) - \alpha^2 n(x_3) = 0.$$

The generic solution of this equation is given by (3.6).

At $\alpha = 0$, one gets a system of algebraic equations for $m, n$ and one differential equation for $\ell$. The generic solutions of these equations are given by (3.6), (3.7).

So, two nontrivial functions $H_\alpha$ and $K_\alpha$ are in the involution on the phase space $T^*S^2$. Moreover, direct calculation yields that they are functionally independent functions on $T^*S^2$. It means that these functions $H_\alpha$ and $K_\alpha$ define an integrable system on the sphere with a quartic in the momenta integral of motion.

If we consider more generic solutions (2.9) of the reflection equations, which depend on four parameters, one gets the same integrals of motion up to rescaling of $x$ and rotations

$$x \to bUx, \qquad J \to UJ, \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{3.8}$$

where $b$ and $\phi$ are the suitable parameters, see [18] for details.

Up to such transformations integrals of motion $H_\alpha$ and $K_\alpha$ depend on five parameters $\alpha, b_0, b_1, c_1/c_2$ and $d$. In [12], another two-parametric family of integrable systems on the sphere with fourth-order integral of motion was defined in implicit form using Lagrangian variables. At present, we do not know whether our system overlaps with this family of systems.

At $\rho \neq 0$, in (2.3), one gets some gyroscopic deformation of the proposed system (3.3). This deformation will be studied in the forthcoming publications.

□

### 3.1. Special cases

In conclusion, let us show some special forms of the Hamiltonian (3.3) in explicit form.

At $\alpha = 0$, the Hamiltonian reads

$$\begin{aligned} H_0 = J_1^2 + J_2^2 + &\left( \frac{x_3^2}{x_3^2 - a^2} - \frac{1}{\left(\ln\left(x_3 + \sqrt{x_3^2 - a^2}\right) + c_3\right)^2} \right) J_3^2 \\ &+ 2b_1c_2 \left( J_1 - \frac{\left(\ln\left(x_3 + \sqrt{x_3^2 - a^2}\right) + c_3\right)x_3 - \sqrt{x_3^2 - a^2}}{\left(x_3^2 - a^2\right)\left(\ln\left(x_3 + \sqrt{x_3^2 - a^2}\right) + c_3\right)} x_1 J_3 \right). \end{aligned} \tag{3.9}$$

It defines a new integrable system on the sphere, which depend on two parameters $b_1c_2$ and $c_3$ only.

If $\alpha = 1$ and $\alpha = 2$, one gets

$$n = c_1 x_3 + c_2\sqrt{a^2 - x_3^2} \qquad \text{and} \qquad n(x_3) = c_1 x_3 \sqrt{a^2 - x_3^2} + c_2\left(a^2 - 2x_3^2\right),$$

respectively. Here, we multiply function $n$ (3.6) on $a^\alpha$ because it is defined up to multiplication on the constant.

Even in these particular cases, the corresponding Hamiltonian $H_\alpha$ (3.3) remains a huge function. We will present it imposing some additional restrictions only.

At $\alpha = 1$ and $c_2 = 0$, the Hamiltonian (3.3) is equal to

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2b_1c_1(J_1x_3 - 2J_3x_1) + 2b_0c_1x_1 + \frac{d}{c_1^2x_3^2} - b_1^2c_1^2a^2. \tag{3.10}$$

After canonical transformation

$$J_1 \to J_1 - c_1b_1x_3, \qquad J_2 \to J_2, \qquad J_3 \to J_3 + c_1b_1x_1, \qquad x_k \to x_k$$

this Hamiltonian $H_1$ (3.10) reads

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2c_1b_0x_1 + c_1^2b_1^2\left(x_1^2 - x_2^2\right) + \frac{d}{c_1^2x_3^2}. \tag{3.11}$$

It is the Hamilton function for the Kowalewski–Chaplygin–Goryaschev top (2.7).

At $\alpha = 1$ and $c_1 = 0$, the corresponding Hamilton function (3.3) is equal to

$$\begin{aligned}\widetilde{H}_1 = J_1^2 + J_2^2 + \frac{2x_3^4 - a^4}{x_3^2\left(x_3^2 - a^2\right)}J_3^2 + 2c_2b_0\frac{x_1x_3}{\sqrt{x_3^2 - a^2}} + \frac{d}{c_2^2\left(x_3^2 - a^2\right)}\\ +2c_2b_1\left(\sqrt{x_3^2 - a^2}J_1 - \frac{2x_3^2 + a^2}{x_3\sqrt{x_3^2 - a^2}}x_1J_3\right) - b_1^2c_2^2a^2.\end{aligned} \tag{3.12}$$

At $\alpha = 2$ and $c_1 = 0$, it has the form

$$\begin{aligned}\widetilde{H}_2 = J_1^2 + J_2^2 + \left(5 + \frac{a^2}{x_3^2}\right)J_3^2 - 4b_0c_2x_3x_1 + \frac{d}{c_2^2\left(2x_3^2 - a^2\right)^2}\\ - 2c_2b_1\left(\left(2x_3^2 - a^2\right)J_1 - \frac{6x_3^2 + a^2}{x_3}x_1J_3\right) + b_1^2c_2^2a^4.\end{aligned} \tag{3.13}$$

The Hamilton functions $H_1$, $\widetilde{H}_1$ and $\widetilde{H}_2$ define particular integrable systems on the sphere, which depend on three parameters only.

## 4. Summary

Using the Lax matrix for the generalized Lagrange system and the standard construction of the commutative subalgebras from the reflection equation theory, we construct a new integrable system on the sphere. The corresponding integrals of motion are given by (3.3); they are second- and fourth-order polynomials in the momenta.

These integrals depend on five parameters $\alpha, b_0, b_1, c_1/c_2$ and $d$ up to canonical transformations. For the special values of parameters, we recover the Kowalevski–Goryachev–Chaplygin system.

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